A MOTIVATED PROOF OF GORDON'S IDENTITIES

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Abstract. We generalize the "motivated proof" of the Rogers-Ramanujan identities given by Andrews and Baxter to provide an analogous "motivated proof" of Gordon's generalization of the Rogers-Ramanujan identities. Our main purpose is to provide insight into certain vertex-algebraic structure being developed.

1. Introduction

In [AB], G. Andrews and A. Baxter have provided an interesting "motivated proof" of the two Rogers-Ramanujan identities (among the large number of proofs in the literature), which we write in the form:

$$\prod_{n\geq 1,\ n\not\equiv 0,\,\pm 2\ (\mathrm{mod}\ 5)} \frac{1}{1-q^n} = \sum_{m\geq 0} p_1(m)q^m$$

and

$$\prod_{n\geq 1,\ n\not\equiv 0,\,\pm 1\ (\mathrm{mod}\ 5)} \frac{1}{1-q^n} = \sum_{m\geq 0} p_2(m)q^m,$$

where

$$p_1(m) = \text{the number of partitions of } m \text{ for which adjacent parts have difference at least 2}$$

and

$$p_2(m) = \text{the number of partitions of } m \text{ for which adjacent parts have difference at least 2 and in which 1 does not appear,}$$

and where q is a formal variable. The idea was to start from the product sides, in which the difference-two condition is invisible, and to both motivate the expressions on the right-hand sides and prove the two identities, as follows:

One subtracts the second product side, called $G_2(q)$, from the first one, called $G_1(q)$, then one divides the difference by q, giving a formal series $G_3(q)$, and then one forms $G_4(q) = (G_2(q) - G_3(q))/q^2$. One repeats this process, giving $G_i(q) = (G_{i-2}(q) - G_{i-1}(q))/q^{i-2}$ for all $i \geq 3$. One notices empirically that for each $i \geq 1$, $G_i(q)$ is a formal power series (that is, it involves only nonnegative powers of q), it has constant term 1, and $G_i(q) - 1$ is divisible by q^i. This is the "Empirical Hypothesis" of Andrews-Baxter. Assuming its truth, one easily gets the two Rogers-Ramanujan identities. Then, with this as motivation, one proceeds to prove the Emprical Hypothesis directly from the product sides, thus proving the Rogers-Ramanujan identities. (In [AB], q is taken to be a complex variable of absolute value less than 1, but in fact, the content of the argument is purely formal, and we shall take q to be a formal variable.)

An initial motivation for the work in [AB] was to show *from the product sides* the highly non-obvious fact that the difference of the two product sides (in the same order as above) is a formal power series with nonnegative coefficients. This was Leon Eherenpreis's problem, and Andrews and Baxter gave a motivated proof of this fact as preparation for their motivated proof of the identities themselves. Also, as is recalled in [AB], that proof of the identities is closely related to Baxter's proof of the identities, starting in [B], and moreover, that proof is also closely related to Rogers's and Ramanujan's proof in [RR].

In the present paper, we generalize the Andrews-Baxter "motivated proof" to give an analogous proof of Gordon's form ([G], [A1]) of the Gordon-Andrews generalizations of the Rogers-Ramanujan identities, essentially in the form in which they are presented in Theorem 7.5 of [A2], where for each $k \geq 2$ and $i = 1, \ldots, k$, a suitable infinite product in q is expressed as a formal power series in q for which the coefficient of q^m is the number of partitions of m such that parts at distance $k-1$ have difference at least 2 and such that 1 appears at most $k-i$ times; the case $k = 2$ is the pair of Rogers-Ramanujan identities. (For a partition $m = p_1 + \cdots + p_n$ of m with $p_1 \geq \cdots \geq p_n > 0$ and for $t \geq 1$, saying that parts at distance t have difference at least 2 means that $p_s - p_{s+t} \geq 2$ whenever $s \geq 1$ and $s + t \leq n$.) We do not address Andrews's multisum form of the sum sides, as presented in Theorem 7.8 of [A2]. While our proof is (necessarily) more complicated than the case $k = 2$, it is similar, although interesting new phenomena arise. Also, since we know what is going to happen, we take the liberty of identifying the appropriate analogue and generalization of the Empirical Hypothesis as our "Empirical Hypothesis," even though we did not observe its validity empirically before actually proving it directly from the product sides. Our proof includes (a variant of) the proof in [AB] as a special case.

Our reason for wanting to work out such a proof stemmed from the relations between the Rogers-Ramanujan identities, and generalizations, and what is now known to be vertex operator algebra theory, as follows:

By retrospective analogy with the approach to the Rogers-Ramanujan identities in [AB], the vertex-operator-theoretic proof of the Rogers-Ramanujan identities along with the vertex-operator-theoretic interpretation of their Gordon-Andrews-Bressoud generalizations in [LW2]–[LW4] also started from the product sides as "given" [LM], and the problem at the time was to discover what turned out to be new structure, not previously anticipated, that would "explain" the sum sides (the difference-two condition and its variants). The result, based on the vertex operator theory whose discovery and development was the subject of those works, was the theory and application of "Z-algebras"—primarily *twisted* Z-algebras in [LW2]–[LW4], and then *untwisted* Z-algebras in [LP],—which later turned out to be understood in retrospect as the natural generating substructures of certain generalized vertex algebras, or abelian intertwining algebras, as developed in [DL], and twisted modules for them. (These Z-algebras, both untwisted and twisted, were also to arise as "parafermion algebras" in conformal field theory ([ZF1], [ZF2]).) In [LW2]–[LW4], each identity was related to certain vertex-operator-theoretic structure constructed from a certain module for an affine Lie algebra, and the structures associated with different identities were not "compared" with one another, in the spirit of product sides being subtracted, etc.; the structures for the different identities were developed in parallel (with the proofs for the parallel structures certainly being closely related). The structures were based on twisted Z-operators, built starting from the twisted vertex operator in [LW1]. Later, a very different vertex-algebraic approach to the sum sides of the Rogers-Ramanujan and Gordon-Andrews identities was developed in [CLM1], [CLM2], [CalLM1] and [CalLM2], this time based

on *untwisted intertwining* operators (in the sense of vertex operator algebra theory), and this time, indeed relating the family of different identities with the same "modulus" (the number $2k+1$ in the notation above). In this work, the classical Rogers-Ramanujan recursion (q-difference equation) and Rogers-Selberg recursions had suggested what turned out to become certain systems of exact sequences constructed from untwisted intertwining operators among the "principal subspaces," in the sense of [FS1]–[FS2], of different modules for certain vertex algebras, and it was this vertex-algebraic structure that was of primary interest.

With this as background, we can now say that the initial reason for our interest in the motivated proof in [AB] is that that proof suggested to one of us (J. L.) and Antun Milas the potential new idea in vertex operator algebra theory to use *twisted* intertwining operators among *twisted* modules for suitable vertex-algebraic structures to develop new structure in the theory that would "re-explain" the identities from this new point of view. Such a program is underway. As was the case in the work mentioned above, the potential new structure suggested, this time, by the "motivated proof" is our main goal.

In Section 2 we give our "motivated proof" of Gordon's identities, in Section 3 we reinterpret the sequence of equalities (2.12) at the core of the proof, and in Section 4 we "explain" the meaning of this sequence of equalities.

The series in q and z below are formal series (rather than convergent series in complex variables).

2. The motivated proof

Fix an integer $k \geq 2$. For each $i = 1, \ldots, k$, define

$$G_i \quad = \prod_{n\geq 1,\; n\not\equiv 0,\, \pm(k+1-i) \pmod{2k+1}} \frac{1}{1-q^n}. \tag{2.1}$$

Recalling the Jacobi triple product identity,

$$\sum_{\lambda\in\mathbb{Z}} (-1)^\lambda z^\lambda q^{\lambda^2} \quad = \quad \prod_{n\geq 0} (1-q^{2n+2})(1-zq^{2n+1})(1-z^{-1}q^{2n+1}),$$

and replacing q by $q^{\frac{2k+1}{2}}$ and z by $q^{\frac{2i-1}{2}}$, we have

$$G_i \quad = \quad \frac{1+\sum_{\lambda\geq 1}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+(k-i+1)\lambda}(1+q^{(2i-1)\lambda})}{\prod_{n\geq 1}(1-q^n)} \tag{2.2}$$

$$\quad = \quad \frac{\sum_{\lambda\geq 0}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+(k+i)\lambda}(1-q^{(k-i+1)(2\lambda+1)})}{\prod_{n\geq 1}(1-q^n)} \tag{2.3}$$

for $i = 1, \ldots, k$.

Define $k-1$ further formal series $G_{k+1}, \ldots, G_{2k-1}$ by

$$G_{k-1+i} \quad = \quad \frac{G_{k-i+1} - G_{k-i+2}}{q^{i-1}} \tag{2.4}$$

for $i = 2, \ldots, k$. Then for these new series, by (2.2) we have

$$\begin{aligned}
G_{k-1+i} &\\
= \quad & \frac{\sum_{\lambda\geq 1}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+i\lambda}(1+q^{(2k-2i+1)\lambda}) - \sum_{\lambda\geq 1}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+(i-1)\lambda}(1+q^{(2k-2i+3)\lambda})}{q^{i-1}\prod_{n\geq 1}(1-q^n)}\\
= \quad & \frac{\sum_{\lambda\geq 1}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+(i-1)\lambda}[q^{\lambda}(1+q^{(2k-2i+1)\lambda}) - (1+q^{(2k-2i+3)\lambda})]}{q^{i-1}\prod_{n\geq 1}(1-q^n)}\\
= \quad & \frac{\sum_{\lambda\geq 1}(-1)^{\lambda+1}q^{(2k+1)\binom{\lambda}{2}+(i-1)\lambda}(1-q^{\lambda})(1-q^{(2k-2i+2)\lambda})}{q^{i-1}\prod_{n\geq 1}(1-q^n)}\\
= \quad & \frac{\sum_{\lambda\geq 0}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+(2k+i)\lambda}(1-q^{\lambda+1})(1-q^{(k-i+1)(2\lambda+2)})}{\prod_{n\geq 1}(1-q^n)} \qquad (2.5)
\end{aligned}$$

for $i = 2, \ldots, k$. In particular, $G_{k-1+i} \in \mathbb{C}[[q]]$ (that is, G_{k-1+i} is a formal power series), and its constant term is 1.

Moreover, (2.5) remains valid for $i = 1$ as well, since the right-hand side for $i = 1$ agrees with (2.3) for $i = k$. Indeed, when $i = 1$, the right-hand side of (2.5) is

$$\frac{\sum_{\lambda\geq 0}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+(2k+1)\lambda}(1-q^{\lambda+1})(1-q^{2k(\lambda+1)})}{\prod_{n\geq 1}(1-q^n)}.$$

Breaking up the last factor in the numerator, we obtain two terms, the first of which can be rewritten as

$$\frac{\sum_{\lambda\geq 0}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+2k\lambda}(q^{\lambda}-q^{2\lambda+1})}{\prod_{n\geq 1}(1-q^n)}.$$

After the re-indexing $\lambda \to \lambda - 1$, the second term becomes

$$\frac{\sum_{\lambda\geq 1}(-1)^{\lambda}q^{(2k+1)\binom{\lambda-1}{2}+(2k+1)(\lambda-1)+2k\lambda}(1-q^{\lambda})}{\prod_{n\geq 1}(1-q^n)}.$$

Noting that $\binom{\lambda-1}{2} + (\lambda - 1) = \binom{\lambda}{2}$ and that allowing $\lambda = 0$ in the numerator only results in adding zero, we can combine the two terms to obtain (2.3) for $i = k$. That is, (2.5) holds for all $i = 1, 2, \ldots, k$.

In general, for $j \geq 1$ and $i = 2, \ldots, k$, define the formal series

$$G_{(k-1)j+i} = \frac{G_{(k-1)(j-1)+k-i+1} - G_{(k-1)(j-1)+k-i+2}}{q^{(i-1)j}}. \qquad (2.6)$$

Theorem 2.1. *For $j \geq 0$ and $i = 1, \ldots, k$, $G_{(k-1)j+i} \in \mathbb{C}[[q]]$ and in fact*

$$\begin{aligned}
G_{(k-1)j+i} &\\
= \quad & \frac{\sum_{\lambda\geq 0}(-1)^{\lambda}q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+i]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(1-q^{(k-i+1)(2\lambda+j+1)})}{\prod_{n\geq 1}(1-q^n)}.
\end{aligned} \qquad (2.7)$$

In particular, denoting the right-hand side of (2.7) by $H_{(k-1)j+i}$, we have that for each $j \geq 1$, the two expressions for G_{kj-j+1} given by (2.7) are equal:

$$H_{(k-1)j+1} \quad = \quad H_{(k-1)(j-1)+k}. \tag{2.8}$$

Proof. By (2.3), (2.7) holds for $j = 0$. By (2.5) and the above, (2.7) and (2.8) both hold for $j = 1$. Take $j \geq 1$. Suppose that (2.7) holds for $G_{(k-1)j+q}$, $1 \leq q \leq k$. We will show that it holds for $G_{(k-1)(j+1)+i}$, $1 \leq i \leq k$.

First let $i = 2, \ldots, k$. By the recursion (2.6), we have

$$\begin{aligned}
G_{(k-1)(j+1)+i} &= \frac{G_{(k-1)j+k-i+1} - G_{(k-1)j+k-i+2}}{q^{(i-1)(j+1)}} \\
&= \frac{\sum_{\lambda\geq 0}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+k-i+1]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(1-q^{i(2\lambda+j+1)})}{q^{(i-1)(j+1)}\prod_{n\geq 1}(1-q^n)} \\
&\quad - \frac{\sum_{\lambda\geq 0}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+k-i+2]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(1-q^{(i-1)(2\lambda+j+1)})}{q^{(i-1)(j+1)}\prod_{n\geq 1}(1-q^n)}
\end{aligned}$$

We use the last factor in each of the two numerators to split each of the two summations into two sums, $\sum_1 = \sum_{11} - \sum_{12}$, $\sum_2 = \sum_{21} - \sum_{22}$, and we combine the terms in a different way: $(\sum_{11} - \sum_{21}) + (-\sum_{12} + \sum_{22})$.

Now $\sum_{11} - \sum_{21}$ equals

$$\frac{\sum_{\lambda\geq 0}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+k-i+1]\lambda}(1-q^{\lambda})(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})}{q^{(i-1)(j+1)}\prod_{n\geq 1}(1-q^n)}. \tag{2.9}$$

Note that the $\lambda = 0$ term in (2.9) vanishes, so the summation is actually over $\lambda \geq 1$. Making the index change $\lambda \to \lambda + 1$ we obtain

$$\begin{aligned}
&\frac{\sum_{\lambda\geq 0}(-1)^{\lambda+1} q^{(2k+1)\binom{\lambda}{2}+[k(j+2)+2k-i+2]\lambda+k(j+1)+k-i+1}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j+1})}{q^{(i-1)(j+1)}\prod_{n\geq 1}(1-q^n)} \\
&= \frac{\sum_{\lambda\geq 0}(-1)^{\lambda+1} q^{(2k+1)\binom{\lambda}{2}+[k(j+2)+i]\lambda+(k-i+1)(j+2)}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j+1})q^{(2k-2i+2)\lambda}}{\prod_{n\geq 1}(1-q^n)}.
\end{aligned} \tag{2.10}$$

Similarly, $-\sum_{12} + \sum_{22}$ equals

$$\begin{aligned}
&\frac{\sum_{\lambda\geq 0}(-1)^{\lambda} q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+k+i]\lambda+(i-1)(j+1)}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(-q^{\lambda+j+1}+1)}{q^{(i-1)(j+1)}\prod_{n\geq 1}(1-q^n)} \\
&= \frac{\sum_{\lambda\geq 0}(-1)^{\lambda} q^{(2k+1)\binom{\lambda}{2}+[k(j+2)+i]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(1-q^{\lambda+j+1})}{\prod_{n\geq 1}(1-q^n)}.
\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we get

$$G_{(k-1)(j+1)+i} = \frac{\sum_{\lambda\geq 0}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+[k(j+2)+i]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j+1})(1-q^{(k-i+1)(2\lambda+j+2)})}{\prod_{n\geq 1}(1-q^n)},$$

proving (2.7) for the case $j+1$ and $i=2,\ldots,k$.

For the case $j+1$ and $i=1$, we observe that (2.7) follows from the induction hypothesis and (2.8) for the case $j+1$, and (2.8) in turn follows (for any j) by virtually the same argument as the one above for $j=1$. □

Theorem 2.1 implies that for $j\geq 0$,

$$\begin{aligned}
&G_{(k-1)j+i}\\
&= \frac{1-q^{(k-i+1)(j+1)}}{(1-q^{j+1})(1-q^{j+2})\cdots}\\
&\quad + \frac{\sum_{\lambda\geq 1}(-1)^\lambda q^{(2k+1)\binom{\lambda}{2}+[k(j+1)+i]\lambda}(1-q^{\lambda+1})\cdots(1-q^{\lambda+j})(1-q^{(k-i+1)(2\lambda+j+1)})}{\prod_{n\geq 1}(1-q^n)}.\\
&= 1+q^{j+1}\gamma_i^{(j+1)}(q) \quad \text{if } 1\leq i\leq k-1\\
\text{or}\quad & 1+q^{j+2}\gamma_k^{(j+2)}(q) \quad \text{if } i=k,
\end{aligned}$$

where

$$\gamma_i^{(j)}(q)\in\mathbb{C}[[q]].$$

This is our "Empirical Hypothesis," in the sense explained in the Introduction.

Using (2.6) in the form (2.17) below together with (2.18) below, we write each G_i, $1\leq i\leq k$, in terms of $G_{(k-1)j+1},\ldots,G_{(k-1)j+k}$ for each $j=0,1,2,\ldots$, giving a sequence of expressions (one for each j) for $G_1,\ldots,G_k$ of the form

$$G_i = {}_ih_1^{(j)}G_{(k-1)j+1}+\cdots+{}_ih_k^{(j)}G_{(k-1)j+k}, \tag{2.12}$$

where for each j the coefficients ${}_ih_l^{(j)}$ form a $k\times k$ matrix $\mathbf{h}^{(j)}$ of polynomials in q with nonnegative integral coefficients. More explicitly, define row vectors

$${}_i\mathbf{h}^{(j)} = [{}_ih_1^{(j)},\ldots,{}_ih_k^{(j)}].$$

For $j=0$ we have

$${}_i\mathbf{h}^{(0)} = [0,\ldots,1,\ldots,0],$$

with 1 in the i-th position, so that $\mathbf{h}^{(0)}$ is the identity matrix. The ${}_i\mathbf{h}^{(j)}$ satisfy the same set of recursions with respect to j, independently of i. Explicitly:

Proposition 2.1. *Let $j\geq 1$. With the left subscript $_i$ suppressed, we have*

$$\begin{aligned}
h_1^{(j)} &= h_1^{(j-1)}+\cdots+h_{k-1}^{(j-1)}+h_k^{(j-1)}\\
h_2^{(j)} &= (h_1^{(j-1)}+\cdots+h_{k-1}^{(j-1)})q^j\\
&\cdots\\
h_{k-1}^{(j)} &= (h_1^{(j-1)}+h_2^{(j-1)})q^{(k-2)j}\\
h_k^{(j)} &= h_1^{(j-1)}q^{(k-1)j}
\end{aligned}$$

or in general,

$$h_l^{(j)} = (h_1^{(j-1)} + \cdots + h_{k-l+1}^{(j-1)})q^{(l-1)j}, \quad 1 \le l \le k. \tag{2.13}$$

In matrix form, this is:

$$\mathbf{h}^{(j)} = \mathbf{h}^{(j-1)}\mathbf{A}_{(j)}, \tag{2.14}$$

with $\mathbf{h}$ *our* $k \times k$ *matrix defined above and with*

$$\mathbf{A}_{(j)} = \begin{bmatrix} 1 & q^j & q^{2j} & \cdots & q^{(k-1)j} \\ \vdots & \vdots & \vdots & \swarrow & \vdots \\ 1 & q^j & q^{2j} & \cdots & 0 \\ 1 & q^j & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.15}$$

In particular,

$$\mathbf{h}^{(j)} = \mathbf{A}_{(1)}\mathbf{A}_{(2)} \cdots \mathbf{A}_{(j)} \tag{2.16}$$

for all $j \ge 0$.

Proof. By (2.6),

$$G_{(k-1)(j-1)+l} = G_{(k-1)(j-1)+l+1} + q^{(k-l)j} G_{(k-1)j+k-l+1} \tag{2.17}$$

for $j \ge 1$, $l = 1, \ldots, k-1$, and the lemma follows from the repeated application of this formula together with the tautological fact that

$$G_{(k-1)(j-1)+k} = G_{(k-1)j+1}. \tag{2.18}$$

□

In the course of the vertex-algebraic interpretation of the Rogers-Selberg recursions in [CLM2], it was implicitly noticed that matrices analogous to the matrices $\mathbf{A}_{(j)}$ along with their inverses, involving the two variables in those recursions, could be used to reformulate those recursions. Such matrices indeed arise naturally from recursions of these types.

Proposition 2.2. *For each* $j \ge 1$ *and* $i, l = 1, \ldots, k$*, the polynomial* ${}_ih_l^{(j)} \in \mathbb{C}[q]$ *is the generating function for partitions with difference at least 2 at distance* $k-1$ *such that* 1 *appears at most* $k-i$ *times, such that the largest part is at most* j*, and such that* j *appears exactly* $l-1$ *times.*

Proof. It is sufficient to show that the combinatorial generating functions described here have the same initial values and recursions as the polynomials ${}_ih_l^{(j)}$. We say that a partition is of type $(k-1, k-i)$ if it has difference at least 2 at distance $k-1$ and 1 appears at most $k-i$ times. Then (2.13) corresponds to the following combinatorial fact: For $j \ge 2$,

the number of partitions of m of type $(k-1, k-i)$ such that the largest part is at most j and such that j appears exactly $l-1$ times

$= \sum_{p=1}^{k-l+1}$ the number of partitions of $m-(l-1)j$ of type $(k-1, k-i)$ such that the largest part is at most $j-1$ and such that $j-1$ appears exactly $p-1$ times.

The initial values

$$ {}_i\mathbf{h}^{(1)} = [1, q, q^2, \cdots, q^{k-i}, 0, \cdots, 0] $$

also match those of the generating functions. □

Recall the products G_i in (2.1).

Theorem 2.2. *For $1 \leq i \leq k$, G_i is the generating function for partitions with difference at least 2 at distance $k-1$ such that 1 appears at most $k-i$ times.*

Proof. This follows immediately from (2.12), Proposition 2.2 and the Empirical Hypothesis. □

This result constitutes Gordon's identities, as formulated in Theorem 7.5 of [A2]; the Rogers-Ramanujan identities form the special case $k = 2$.

3. Matrix interpretation

The right-hand side of (2.12) suggests a product of matrices, and the recursions for the ${}_ih_l^{(j)}$ come from the recursions (2.17) (or equivalently, (2.6)) for the G_s, $s \geq 1$, together with (2.18). We now express all of this in matrix form, and in the process we quickly rederive Proposition 2.1, obtaining the matrices $\mathbf{h}^{(j)}$ and their properties from (2.17) and (2.18).

Set

$$ \mathbf{G}_{(0)} = \begin{bmatrix} G_1 \\ \vdots \\ G_k \end{bmatrix} $$

and in general,

$$ \mathbf{G}_{(j)} = \begin{bmatrix} G_{(k-1)j+1} \\ \vdots \\ G_{(k-1)j+k} \end{bmatrix} $$

for $j \geq 0$. Also set

$$ \mathbf{B}_{(j)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & q^{-j} & -q^{-j} \\ 0 & 0 & \cdots & q^{-2j} & -q^{-2j} & 0 \\ \vdots & \vdots & \swarrow & \vdots & \vdots & \vdots \\ 0 & q^{-(k-2)j} & \cdots & 0 & 0 & 0 \\ q^{-(k-1)j} & -q^{-(k-1)j} & \cdots & 0 & 0 & 0 \end{bmatrix} $$

for $j \geq 1$. Then (2.6)) (or equivalently, (2.17)) and (2.18) assert that

$$ \mathbf{G}_{(j)} = \mathbf{B}_{(j)}\mathbf{G}_{(j-1)} \tag{3.1} $$

for $j \geq 1$, so that

$$ \mathbf{G}_{(j)} = \mathbf{B}_{(j)}\mathbf{B}_{(j-1)} \cdots \mathbf{B}_{(1)}\mathbf{G}_{(0)} $$

for $j \geq 0$. But

$$ \mathbf{B}_{(j)} = (\mathbf{A}_{(j)})^{-1} $$

(recall (2.15)), which gives

$$ \mathbf{A}_{(j)}\mathbf{G}_{(j)} = \mathbf{G}_{(j-1)} $$

for $j \geq 1$ and

$$\mathbf{G}_{(0)} = \mathbf{A}_{(1)}\mathbf{A}_{(2)} \cdots \mathbf{A}_{(j)}\mathbf{G}_{(j)}$$

for $j \geq 0$. Defining $\mathbf{h}^{(j)}$ recursively by

$$\mathbf{h}^{(0)} = \text{identity matrix},$$

$$\mathbf{h}^{(j)} = \mathbf{h}^{(j-1)}\mathbf{A}_{(j)}$$

for $j \geq 1$ (cf. (2.14)), we have (2.16) along with (2.12), in the form

$$\mathbf{G}_{(0)} = \mathbf{h}^{(j)}\mathbf{G}_{(j)}$$

for each $j \geq 0$. Thus from (2.17) and (2.18) expressed in matrix form, we have an "automatic" reformulation and proof of Proposition 2.1, along with (2.12).

The combinatorial interpretation of the entries of $\mathbf{h}^{(j)}$ in Proposition 2.2 is a separate matter, as is the combinatorial interpretation of the G_s, $s \geq 1$, given in Theorem 4.1 below.

4. Interpretation of the sequence of expressions for G_i

All of the formal power series G_s for $s \geq 1$ can be interpreted as combinatorial generating functions, and this will allow us to "explain" the meaning of the sequence of equalities (2.12) for $G_1, \ldots, G_k$.

We start with the expression of $G_1, \ldots, G_k$ as the generating functions given by Theorem 2.2. The recursions (2.17), or equivalently, (2.6), determine all the G_s for $s \geq 1$ from $G_1, \ldots, G_k$, and we know from Theorem 2.1 that all the G_s, $s \geq 1$, are formal power series.

We have the following "complement" to Proposition 2.2, reflecting and illustrating the complementary nature of the recursions (2.13) and (2.17) (or equivalently, of (2.14) and (3.1)):

Theorem 4.1. *For $j \geq 0$ and $l = 1, \ldots, k$, the formal power series $G_{(k-1)j+l}$ is the generating function for partitions with difference at least 2 at distance $k-1$ such that the smallest part is greater than j and such that $j+1$ appears at most $k-l$ times.*

Proof. It is sufficient to show that these combinatorial generating functions have the same initial values and recursions as the formal power series $G_{(k-1)j+l}$. The recursion (2.17) for $j \geq 1$ and $l = 1, \ldots, k-1$ corresponds to the following combinatorial fact: For $j \geq 1$ and $l = 1, \ldots, k-1$,

the number of partitions of m with difference at least 2 at distance $k-1$ such that the smallest part is at least j and such that j appears exactly $k-l$ times

$=$ the number of partitions of $m-(k-l)j$ with difference at least 2 at distance $k-1$ such that the smallest part is greater than j and such that $j+1$ appears at most $l-1$ times.

By Theorem 2.2 (the case $j=0$), these assertions prove the result. (Note that the combinatorial interpretations of the two expressions equated in (2.18) indeed agree.) □

For $j = 0$, (2.12) says simply that $G_i = G_i$, and for $j \geq 1$, combining Proposition 2.2 and Theorem 4.1 we immediately have:

Theorem 4.2. *For $l = 1, \ldots, k$ and $j \geq 1$, the right-hand side of (2.12) expresses the generating function G_i as the sum of its contributions corresponding to the number of times, namely, $0, 1, \ldots, k-1$, that the part j appears in a partition.*

Remark 4.1. With the benefit of the picture that has emerged, we can give an alternate, shorter proof of Theorem 2.2 (Gordon's identities), without needing Proposition 2.1 or 2.2, using the following uniqueness observation: Let $J_1, J_2, \dots$ be a sequence of formal power series in q with constant term 1 satisfying the recursions (2.17) for $j \geq 1$ and $l = 1, \dots, k-1$ (with J in place of G), and suppose that the Empirical Hypothesis holds for $J_1, J_2, \dots$. The comments above in connection with (2.12) give a sequence of expressions of the form (2.12) (with J in place of G), with the coefficients the same polynomials ${}_ih_l^{(j)}$ as in (2.12) (and now, we do not have to compute them). By the Empirical Hypothesis, the k formal power series $J_1, \dots, J_k$ are uniquely determined, and thus so is the whole sequence $J_1, J_2, \dots$. But by the proof of Theorem 4.1, the combinatorial generating functions defined in the statement of Theorem 4.1 form a sequence $K_1, K_2, \dots$ of formal power series with constant term 1 satisfying the recursions (2.17), and the Empirical Hypothesis trivially holds for $K_1, K_2, \dots$. Thus by the uniqueness, $J_s = K_s$ for each $s \geq 1$. Then by Theorem 2.1, which gives the Empirical Hypothesis for $G_1, G_2, \dots$, we have (without using Theorem 2.2) that $G_s = K_s$ for each $s \geq 1$, and this statement for $s = 1, \dots, k$ constitutes Gordon's identities. This remark generalizes the corresponding alternate proof of the Rogers-Ramanujan identities discsussed in [AB], [R] and [A3].

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